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LETTER TO THE EDITOR

Lie symmetries for the charge–monopole problem

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Abstract. In this letter we obtain the Lie symmetries for the equation of motion of an electric charge interacting with a magnetic monopole, fixed at the origin. We also discuss the construction of first integrals for this problem.

In recent years there has been a revival of interest in the analysis of the continuous symmetries of differential equations (the so-called Lie symmetries), especially for the nonlinear wave equations. The knowledge of these symmetries is an important step in finding particular solutions of nonlinear equations and for the identification of conservation laws (Ovsjannikov 1982, Bluman and Cole 1974). Several papers in this journal have been dedicated also to discuss the Lie symmetries for some important discrete physical systems described by ordinary differential equations: Wulfman and Wybourne (1976) found the Lie symmetry group for the simple harmonic oscillator; Prince and Eliezer (1980, 1981) did the same for the time-dependent oscillator and for the Kepler problem. Similar problems have been considered by other authors (Leach 1981, Moreira 1983).

Here we consider the equation of motion:

$$\ddot{\boldsymbol{r}} = \mu \frac{\boldsymbol{v} \times \boldsymbol{r}}{r^3} \tag{1}$$

where $\mu = eg/mc$, that describes the motion of an electric charge e in the field of a magnetic monopole g, fixed at the origin.

The classical solution of this equation was established by Poincaré (1896); he showed that the trajectory of the charge is over a cone with the apex at the origin. The direction of the axis of the cone is given by the generalised angular momentum:

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{p} - \frac{eg}{c} \frac{\boldsymbol{r}}{r}. \tag{2}$$

Dirac (1931) has considered this problem from a quantum mechanical point of view and has found the important relation $eg = \frac{1}{2}\mu\hbar$, μ being an integer, concerning the quantisation of the electric and magnetic charge.

The general conditions (Lie 1896) for the invariance of an equation

$$\Lambda_i = \ddot{x}_i - f_i(\dot{x}, x, t) = 0 \tag{3}$$

under the infinitesimal transformations

$$x_i \rightarrow x_i' = x_i + \varepsilon\eta_i(x, t)$$
$$t \rightarrow t' = t + \varepsilon\xi(x, t) \tag{4}$$

are

$$U''\Lambda_i = 0 \tag{5}$$

where the second extended operator U'' is given by

$$U'' = \xi\frac{\partial}{\partial t} + \eta_i\frac{\partial}{\partial x_i} + \eta_i'\frac{\partial}{\partial \dot{x}_i} + \eta_i''\frac{\partial}{\partial \ddot{x}_i} \tag{6}$$

with

$$\eta_1' = \frac{\mathrm{d}\eta_i}{\mathrm{d}t} - \dot{x}_i\frac{\mathrm{d}\xi}{\mathrm{d}t}$$
$$\eta_i'' = \frac{\mathrm{d}\eta_i'}{\mathrm{d}t} - \ddot{x}_i\frac{\mathrm{d}\xi}{\mathrm{d}t}. \tag{7}$$

The application of the conditions (5) to equation (1) leads to a system of 36 linear partial differential equations whose solution leads to the following independent generators:

$$X_1 = \frac{\partial}{\partial t}$$
$$X_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$
$$X_3 = t^2\frac{\partial}{\partial t} + t\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$
$$X_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \tag{8}$$
$$X_5 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}$$
$$X_6 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}.$$

The Lie algebra associated with these operators is given by the commutators:

$$[X_1, \tfrac{1}{2}X_2] = X_1 \qquad [\tfrac{1}{2}X_2, X_3] = X_3 \qquad [X_1, X_3] = X_2$$
$$[X_4, X_5] = X_6 \qquad [X_4, X_6] = X_5 \qquad [X_5, X_6] = -X_4 \tag{9}$$
$$[X_i, X_j] = 0 \qquad \text{for the other cases.}$$

This algebra is a direct sum of the subalgebra $\{X_1, X_2, X_3\}$ that corresponds to the SO(2, 1) group and of the subalgebra $\{X_4, X_5, X_6\}$ corresponding to the SO(3) group. Therefore SO(2, 1) × SO(3) is the Lie symmetry group for the equation (1).

We can find first integrals for the equation (1) from the knowledge of the Lie symmetries. A direct way to do this is to apply the first extended operator (Lutzky 1979)

$$U' = \xi \frac{\partial}{\partial t} + \eta_i \frac{\partial}{\partial x_i} + \eta_i' \frac{\partial}{\partial \dot{x}_i} \tag{10}$$

to a first integral to get a new constant:

$$U' I_1 = I_2. \tag{11}$$

In our case if we start from the conservation of the kinetic energy $I_1 = \frac{1}{2} m \boldsymbol{v}^2$ (obviously conserved for a force of this kind) we get:

$$\begin{aligned} &U'_{X_1} I_1 = 0 \qquad U'_{X_2} I_1 = -2I_1 \\ &U'_{X_3} I_1 = \boldsymbol{r} \cdot \boldsymbol{v} - t\boldsymbol{v}^2 = I_2. \end{aligned} \tag{12}$$

If we apply again these extended operators to I_2, we get

$$\begin{aligned} &U'_{X_1} I_2 = -2I_1 \qquad U'_{X_2} I_2 = 0 \\ &U'_{X_3} I_2 = (\boldsymbol{r} - \boldsymbol{v}t)^2 = I_3. \end{aligned} \tag{13}$$

Similarly the components of the generalised angular momentum $\boldsymbol{\tau}$ can be obtained by applying U'_{X_3, X_4, X_5} to any component of this vector. With these constants of motion the trajectory of the charge can be completely determined.

An alternative method introduced by Prince and Eliezer (1981) permits the construction of first integrals starting from the symmetry generators. This procedure could be used also in this case to determine (2), (12) and (13).

The analysis of the symmetries and the construction of the first integrals for the charge-monopole problem can be made by several approaches. Jackiw (1980) applied the Noether theorem to a Lagrangian with a singular potential to get the dynamical symmetry group [SO(3) × SO(2, 1)]; he has used symmetry transformations with linear dependence in velocity. Another possibility would be to try a generalisation of the Noether theorem within the formulation of Wu and Yang (1976) for the charge-monopole interaction. In order to circumvent the singularity problem they divided the space outside the monopole into two overlapping regions R_a and R_b and defined singularity-free electromagnetic potentials in R_a and R_b. In a third approach to this problem we can use the formalism introduced by Sokolov (1976): he wrote a Lagrangian in a four-dimensional space (where the coordinates are the three Euler angles and the separation between the charge and the monopole), and with this procedure he avoids the difficulties arising from the singularity of the potential. A direct application of the Noether theorem to this Lagrangian gives us the same symmetry transformations and the first integrals I_1, I_2 and I_3. The approach used here by applying the Lie symmetries avoids the utilisation of singular potentials, of a multiple-valued action integral or the introduction of additional variables. Of course in the quantisation of the system we must make use of one of these Hamiltonian approaches.

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